Volume 4, Issue 12, December 2014 ISSN: 2277 128X
International Journal of Advanced Research in Computer Science and Software Engineering
Research Paper
Available online at: www.ijarcsse.com

# Quartic Spline Collocation for Solving Eigenvalue Problems

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***Abstract—In this paper, we develop quartic spline collocation methods and treat a number of eigenvalue problems defined by partial differential equations with constant and variable coefficients, on rectangular or circular domains. The presented methods are an interesting and easy to implement on a computer and being capable of producing very accurate approximations of eigenvalue defined by partial eigenvalue problems.***

***Keywords— Spline collocation, partial differential equations, eigenvalue problems.***

## I. INTRODUCTION

This paper solves a class of eigenvalue problems defined by partial differential equations with constant and variable coefficients, on rectangular or circular domain. The methods [1-3] are based on quartic splines which provided by Christina [4] for linear second order elliptic partial differential equations (PDEs), that is, piecewise quartic polynomials with $C^3$ continuity, and the collocation discretization methodology with the midpoints of a uniform partition being the collocation points [5-6]. The choice of approximating space, basis functions and collocation points plays an important role in the accuracy of the approximation and the efficiency of the calculations. Spline collocation methods are discussed in many papers [7,8,9,11,12], and it is quite common in these literature, to pick as data points the gridpoints of the partition, when odd degree splines are used, and the midpoints when even degree splines are used. In this paper, quartic spline collocation methods [6] are applied to solve class of eigenvalue problems which solved by many others methods [1,2,3,4,5].

Let $\overline{\Omega}=(0,a)\times(0,b)$ be a rectangular domain and let $\partial\Omega$ be its boundary. In this paper, we shall assume that $\overline{\Omega}$ is the unit square without loss of generality, namely, that $a=b=1$. Let us consider in $\overline{\Omega}$ the fixed membrane problem defined by the PDE:

(1.1) $$u_{xx}(x,y)+u_{yy}(x,y)+\lambda\, u(x,y)=0$$

with Dirichlet boundary conditions

(1.2) $$u(x,y)=0 \quad \text{on } \partial\Omega \equiv \text{boundary of } \overline{\Omega}$$

The eigenvalues of problem (1.1) and (1.2) are known to be

(1.3) $$\lambda_{rs}=\pi^2[(\frac{r}{a})^2+(\frac{s}{b})^2], \qquad r,s=1,2,3,\ldots$$

and the corresponding eigenfunctions are

(1.4) $$u_{rs}(x,y)=C_{rs}\sin(\frac{r\pi x}{a})\sin(\frac{r\pi y}{b}).$$

The boundary shape has been transformed to the unit square $\overline{\Omega}\equiv\{(0,1)\times(0,1)\}$. Let us define on the intervals $0\le x\le 1$ and $0\le y\le 1$ the partitions:

$$\Delta_x \equiv \{0=x_0<x_1<\ldots<x_M=1\}$$
$$\Delta_y \equiv \{0=y_0<y_1<\ldots<y_N=1\}$$

and the set of points

(1.5)
$$T^x \equiv \{\tau_i^x=\frac{x_{i-1}+x_i}{2},\ i=1,2,\ldots,M,\ x_0=\tau_0^x,\ x_M=\tau_{M+1}^x\}$$
$$T^y \equiv \{\tau_j^y=\frac{y_{j-1}+y_j}{2},\ j=1,2,\ldots,N,\ y_0=\tau_0^y, y_N=\tau_{N+1}^y\}$$

the midpoints of the partitions $\Delta_x$ and $\Delta_y$, respectively. Then $T\equiv\{(\tau_i^x,\tau_j^y),$ $i=0,1,\ldots,M+1,\ \ j=0,1,\ldots,N+1\}$ is the set collocation points $\overline{\Omega}$.

## II. METHOD OF SOLUTION

In this section, we present quartic spline interpolation with uniform grid partitions $\Delta_x$ and $\Delta_y$ of the intervals $[0,a],\ [0,b]$ with mesh sizes $h_x = \dfrac{a}{M},\ h_y = \dfrac{b}{N}$ respectively. Then $\Delta \equiv \Delta_x \times \Delta_y$ is the induced grid partition of $\bar{\Omega} \equiv \Omega \cup \partial\Omega \equiv [0,a]\times[0,b]$, and $T \equiv \{(\tau_i^x, \tau_j^y),\quad i = 0,1,\ldots,M+1,\ \ j = 0,1,\ldots,N+1\}$ is the set collocation points $\overline{\Omega}$.

Let $S_{4,\Delta} \equiv S_{4,\Delta_x} \otimes S_{4,\Delta_y}$ be the space of piecewise biquartic splines with respect to partition $\Delta$ of $\overline{\Omega}$, where $S_{4,\Delta_x}$ and $S_{4,\Delta_y}$ are the space of quartic splines with respect to partitions $\Delta_x, \Delta_y$, respectively. Based on partition $\Delta$, we define the biquartic spline space, the space of biquartic piecewise polynomials of continuity $C^3(\overline{\Omega})$ on the nodes of partition $\Delta$, and a set of basic functions for it. We choose the sets $\{\Phi_i^x(x)\}_{i=-1}^{M+2}$ and $\{\Phi_j^y(y)\}_{j=-1}^{N+2}$ as the basis functions for $S_{4,\Delta_x}$ and $S_{4,\Delta_y}$ respectively, where $\Phi_i^x(x) = \Phi(\dfrac{x}{h_x} - i + 3),\ i = -1,0,\ldots,M+2$ and $\Phi_j^y(y) = \Phi(\dfrac{y}{h_y} - j + 3),\quad j = -1,0,\ldots,N+2$, and the quartic spline function $\Phi$ is defined by, [6]

$$\Phi(x) = \begin{cases} x^4 & \text{for } 0 \le x \le 1 \\ x^4 - 5(x-1)^4 & \text{for } 1 \le x \le 2 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 & \text{for } 2 \le x \le 3 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 & \text{for } 3 \le x \le 4 \\ x^4 - 5(x-1)^4 + 10(x-2)^4 - 10(x-3)^4 + 5(x-4)^4 & \text{for } 4 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases} \tag{2.1}$$

A set of basis functions for the biquartic spline space defined with respect to partition $\Delta$ is the tensor product $\Phi \equiv \{\Phi_i^x(x)\Phi_j^y(y)\}_{i=-1}^{M+2}\,{}_{j=-1}^{N+2}$ of quartic B-splines.

Let $S \in S_{4,\Delta}$ be the biquartic spline interpolant of the true solution $u$ defined by the interpolation relations

$$S(\tau_i^x, \tau_j^y) = u(\tau_i^x, \tau_j^y); \quad 0 \le i \le M+1,\ \ 0 \le j \le N+1. \tag{2.2}$$

where the approximation of $S$ can be written as

$$S = \sum_{i=-1}^{M+2} \sum_{j=-1}^{N+2} \theta_{ij} \Phi_i(x) \Phi_j(y) \tag{2.3}$$

Whenever the boundary conditions (1.2) of the problem are homogeneous Dirichlet or Neuman, that is $u = 0$ or $u_n = 0$, on each of the boundary subintervals of partition $\Delta$ of $\Omega$, assume that the approximate space satisfies exactly the boundary conditions. A basis for such a space is the tensor product of the sets $\{\tilde{\Phi}_i(x)\}_{i=1}^{M}$ and $\{\tilde{\Phi}_j(y)\}_{j=1}^{N}$ where

$$\tilde{\Phi}_1 = \Phi_1 \pm \Phi_0, \qquad \tilde{\Phi}_2 = \Phi_2 \pm \Phi_{-1}$$

$$\tilde{\Phi}_i = \Phi_i, \qquad i = 3,\ldots,M-2$$

$$\tilde{\Phi}_{M-1} = \Phi_{M-1} \pm \Phi_{M+2}, \qquad \tilde{\Phi}_M = \Phi_M \pm \Phi_{M+1}$$

and $\{\tilde{\Phi}_j(y)\}_{j=1}^{N}$ are defined in a similar way. The sign in the definition of $\tilde{\Phi}_i$ is chosen according to the type of boundary conditions. The "-"corresponds to Dirichlet conditions, while the "+" corresponds to Neumann conditions.

Then the approximation (2.3) can be written as

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} \tilde{\theta}_{ij} \tilde{\Phi}_i(x) \tilde{\Phi}_j(y) \tag{2.4}$$

## III. FIXED MEMBRANE PROBLEM

To approximate fixed membrane problem defined by equations (1.1) with supplementary condition

$$u(x,0) = u(x,1) = 0 \text{ and } u(0,y) = u(1,y) = 0 \tag{3.1}$$

we shall use biquartic spline collocation methods defined by the approximation (2.4) and the quartic spline function (2.1) to get the following relations

$$u_{xx}(\tau_l^x,\tau_k^y)=\sum_{i=1}^{M}\sum_{j=1}^{N}\tilde{\theta}_{ij}\tilde{\Phi}_i^{\backslash\backslash}(\tau_l^x)\tilde{\Phi}_j(\tau_k^y) \tag{3.2}$$

$$u_{yy}(\tau_l^x,\tau_k^y)=\sum_{i=1}^{M}\sum_{j=1}^{N}\tilde{\theta}_{ij}\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j^{\backslash\backslash}(\tau_k^y) \tag{3.3}$$

$$u(\tau_l^x,\tau_k^y)=\sum_{i=1}^{M}\sum_{j=1}^{N}\tilde{\theta}_{ij}\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j(\tau_k^y) \tag{3.4}$$

where $l=1,2,\ldots,M,\quad k=1,2,\ldots,N$.

Then by using the above approximations (3.2)-(3.4), equation (1.1) is transformed to the following eigenvalue problem

$$A\tilde{\theta}+\lambda D\tilde{\theta}=0 \tag{3.5}$$

where

$$A=\tilde{\Phi}_i^{\backslash\backslash}(\tau_l^x)\tilde{\Phi}_j(\tau_k^y)+\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j^{\backslash\backslash}(\tau_k^y)$$
$$D=\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j(\tau_k^y),$$

$l=1,2,\ldots,M,\quad k=1,2,\ldots,N$ and $\tilde{\theta}=(\tilde{\theta}_1,\tilde{\theta}_2,\ldots,\tilde{\theta}_{MN})^T$.

The matrices $A$ and $D$ can be written as the product of the matrices as the following

$$A=C.Q+P.H \tag{3.6}$$

$$D=P.Q \tag{3.7}$$

where

$$p=\frac{1}{24\times16}\begin{bmatrix}154I & 75I & I & & & & & & \\ 75I & 230I & 76I & I & & & & & \\ I & 76I & 230I & 76I & I & & & & \\ & I & 76I & 230I & 76I & I & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & & & & I & 76I & 230I & 76I & I \\ & & & & & I & 76I & 230I & 75I \\ & & & & & & I & 75I & 154I\end{bmatrix} \tag{3.8}$$

$$Q=\begin{bmatrix}T & & & & & \\ & T & & & & \\ & & . & & & \\ & & & . & & \\ & & & & T & \\ & & & & & T\end{bmatrix} \tag{3.9}$$

$$T=\frac{1}{24\times16}\begin{bmatrix}154 & 75 & 1 & & & & & & \\ 75 & 230 & 76 & 1 & & & & & \\ 1 & 76 & 230 & 76 & 1 & & & & \\ & 1 & 76 & 230 & 76 & 1 & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & & & & 1 & 76 & 230 & 76 & 1 \\ & & & & & 1 & 76 & 230 & 75 \\ & & & & & & 1 & 75 & 154\end{bmatrix}$$

 

(3.10) $$C = \frac{1}{8h^2}\begin{bmatrix} -14I & 3I & I & & & & & & \\ 3I & -10I & 4I & I & & & & & \\ I & 4I & -10I & 4I & I & & & & \\ & I & 4I & -10I & 4I & I & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & & & & I & 4I & -10I & 4I & I \\ & & & & & I & 4I & -10I & 3I \\ & & & & & & I & 3I & -14I \end{bmatrix}$$

(3.11) $$H = \begin{bmatrix} T_1 & & & & & \\ & T_1 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & T_1 & \\ & & & & & T_1 \end{bmatrix}$$

where

$$T_1 = \frac{1}{8h^2}\begin{bmatrix} -14 & 3 & 1 & & & & & & \\ 3 & -10 & 4 & 1 & & & & & \\ 1 & 4 & -10 & 4 & 1 & & & & \\ & 1 & 4 & -10 & 4 & 1 & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & & & & 1 & 4 & -10 & 4 & 1 \\ & & & & & 1 & 4 & -10 & 3 \\ & & & & & & 1 & 3 & -14 \end{bmatrix}$$

and $I$ is the identity matrix of order $M$ and $N$, respectively.

The main advantage of using this method over Liu and Ortiz [2], El-Hawary [3,4,5,6] is that for certain values of $M$ and $N$ the elements of the above matrices can be evaluated once and for all. Economization in computation will be achieved if, for example, and the above matrices are stored for different values of $M$ and $N$.

## IV. CLASS OF SYMMETRIES OF EIGEN FUNCTIONS

We shall now consider the use of symmetries in a given differential eigenvalue problem to try to reduce the size of associated linear algebraic eigenvalue problem. This discussion will initially be referred to the vibrating membrane problem, (1.1) and (1.2) as given in Liu and Ortiz [2] and in El-Hawary [10].

Let us assume that $r$ is the number of interior lines in the domain $\overline{\Omega}$, and it seems natural to treat symmetrical problems by using an even number of interior lines, we choose $r = 2m$.

We shall consider first the even-even symmetry class. We have

(4.1) $$u_{2m+1-k}(x) = u_k(x), \qquad k = 1,2,\ldots,m$$

## V. HELMHOLTZ EIGENVALUE PROBLEM

Let us consider the eigenvalue problem defined by the helmholtz partial differential equations over a unit circular domain $C$ with boundary $B$:

(5.1) $$\nabla u(r,\theta) + \lambda u(r,\theta) = 0 \quad \text{on } C$$

and

(5.2) $$\begin{aligned} u(0,\theta) = u(1,\theta) = 0, &\qquad \theta \in (0,2\pi) \\ u(r,0) = u(r,1) = 0, &\qquad r \in (0,1) \end{aligned}$$

where $\nabla$ stands for the laplacian operator coordinates,

(5.3) $$\nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

making use of the transformation

$$x = r, \qquad y = \frac{\theta}{2\pi}$$

the circular domain $C$ is mapped into the square domain $[0,1]\times[0,1]$ and the problem expressed by eqs. (5.1), (5.2) is transformed to

(5.4) $$u_{xx}(x,y) + \frac{1}{x}u_x(x,y) + \frac{1}{x^2}\left(\frac{1}{4\pi^2}\right)u_{yy}(x,y) + \lambda u(x,y) = 0$$

with the boundary conditions

(5.5) $$\begin{array}{ll} u(0,y) = u(1,y) = 0, & y \in (0,1) \\ u(x,0) = u(x,1) = 0, & r \in (0,1) \end{array}.$$

To solve the problems (5.4) and (5.5) we set

(5.6) $$u_{xx}(\tau_l^x, \tau_k^y) = \sum_{i=1}^{M}\sum_{j=1}^{N} \tilde{\theta}_{ij}\tilde{\Phi}_i^{\backslash\backslash}(\tau_l^x)\tilde{\Phi}_j(\tau_k^y)$$

(5.7) $$u_x(\tau_l^x, \tau_k^y) = \sum_{i=1}^{M}\sum_{j=1}^{N} \tilde{\theta}_{ij}\tilde{\Phi}_i^{\backslash}(\tau_l^x)\tilde{\Phi}_j(\tau_k^y)$$

(5.8) $$u_{yy}(\tau_l^x, \tau_k^y) = \sum_{i=1}^{M}\sum_{j=1}^{N} \tilde{\theta}_{ij}\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j^{\backslash\backslash}(\tau_k^y)$$

(5.9) $$u(\tau_l^x, \tau_k^y) = \sum_{i=1}^{M}\sum_{j=1}^{N} \tilde{\theta}_{ij}\tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j(\tau_k^y)$$

where $\tau_l^x \in T^x, \quad l = 1,2,\ldots,M, \quad \tau_k^y \in T^y, \quad k = 1,2,\ldots,N$ defined by equation (1.5). Using the above approximations defined by equations (5.6)-(5.9), equation (5.4) is transformed to the following eigenvalue problem

(5.10) $$A\,\tilde{\theta} + \lambda D\tilde{\theta} = 0$$

where

$$A = \tilde{\Phi}_i^{\backslash\backslash}(\tau_l^x).\,\tilde{\Phi}_j(\tau_k^y) + A_1[\tilde{\Phi}_i^{\backslash}(\tau_l^x)\,.\tilde{\Phi}_j(\tau_k^y)] + A_2[\tilde{\Phi}_i(\tau_l^x)\,.\tilde{\Phi}_j^{\backslash\backslash}(\tau_k^y)]$$

$$D = \tilde{\Phi}_i(\tau_l^x)\tilde{\Phi}_j(\tau_k^y)$$

$l = 1,2,\ldots,M, \quad k = 1,2,\ldots,N$ and $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_{MN})^T$.

The matrices $A$ and $D$ can be written as the product of the matrices as the following

(5.11) $$A = C.Q + A_1.(R.Q) + A_2.(P.H)$$

(5.12) $$D = P.Q$$

where the matrices $C, Q, P,$ and $H$ are defined above by relations (3.8)-(3.11) and the matrices $R, A_1$ and $A_2$ are defined as the following:

$$R = \frac{1}{48h^2}\begin{bmatrix} 22I & 23I & I & & & & & & \\ -21I & 0 & 22I & I & & & & & \\ -I & -22I & 0 & 22I & I & & & & \\ & -I & -22I & 0 & 22I & I & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & & & & -I & -22I & 0 & 22I & I \\ & & & & & -I & -22I & 0 & 21I \\ & & & & & & -I & -23I & -22I \end{bmatrix}$$

and $A_1, A_2$ are block diagonal matrices in the form:

 

$$A_j = \begin{bmatrix} A_j^1 & & & & \\ & A_j^2 & & & \\ & & . & & \\ & & & . & \\ & & & & A_j^M \end{bmatrix}, \qquad j = 1,2$$

where

$$A_1^{(k)} = [a_{lk}^{(1)}\,\delta_{lj}], \qquad a_{lk}^{(1)} = \frac{1}{\tau_l^x}$$

$$A_2^{(k)} = [a_{lk}^{(2)}\,\delta_{lj}], \qquad a_{lk}^{(2)} = \frac{1}{(2\pi\,\tau_l^x)^2}$$

and

$$\tau_l^x = \frac{x_l + x_{l-1}}{2}, \qquad l = 1,2,\ldots,M, \quad k = 1,2,\ldots,N .$$

## VI. EIGENVALUE PROBLEM FOR PDE WITH VARIABLE COEFFICIENTS

Let us consider, an eigenvalue problem defined on a unit square domain R [2], with the Dirichlet boundary conditions:

(6.1) $$u_{xx}(x,y) + u_{yy}(x,y) - 10\,x\,\sin(3\pi\,y)\,u(x,y) + \lambda u(x,y) = 0$$

with

(6.2) $$u(x,y) = 0 \quad \text{on } B$$

As before we define on $(x,y) \in [(0,1)\times(0,1)]$ the partitions:

$$\Delta_x \equiv \{0 = x_0 < x_1 < \ldots < x_M = 1\}$$

$$\Delta_y \equiv \{0 = y_0 < y_1 < \ldots < y_N = 1\}$$

To solve the problems (6.1), (6.2) we use the approximations (3.2)-(3.4) with the midpoints $T \equiv \{(\tau_i^x, \tau_j^y),$ $i = 0,1,\ldots,M+1\,,\;\; j = 0,1,\ldots,N+1\,\}$ as a collocation points of $C$, where

(6.3)

$$T^x \equiv \{\tau_i^x = \frac{x_{i-1} + x_i}{2},\; i = 1,2,\ldots,M,\; x_0 = \tau_0^x,\; x_M = \tau_{M+1}^x\}$$

$$T^y \equiv \{\tau_j^y = \frac{y_{j-1} + y_j}{2},\; j = 1,2,\ldots,N,\; y_0 = \tau_0^y, y_N = \tau_{N+1}^y\}$$

Then equation (6.1) is transformed to the following eigenvalue problem

(6.4) $$A\,\tilde{\theta} + \lambda D\tilde{\theta} = 0$$

where $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_{MN})^T$,

$$A = \tilde{\Phi}_i^{\backslash\backslash}(\tau_l^x).\,\tilde{\Phi}_j(\tau_k^y) + \tilde{\Phi}_i(\tau_l^x)\,.\tilde{\Phi}_j^{\backslash\backslash}(\tau_k^y) + A_1[\tilde{\Phi}_i(\tau_l^x)\,.\tilde{\Phi}_j(\tau_k^y)]$$

$$D = \tilde{\Phi}_i(\tau_l^x)\,\tilde{\Phi}_j(\tau_k^y)$$

and $A_1$ is block diagonal matrix in the form

(6.5) $$A_1 = \begin{bmatrix} A^1 & & & & \\ & A^2 & & & \\ & & . & & \\ & & & . & \\ & & & & A^M \end{bmatrix},$$

where

$$A^{(k)} = [a_{lj}^{(k)}] = [a_{lk}\,\delta_{lj}], \qquad a_{lk} = -10\,\tau_l^x\,\sin(3\pi\,\tau_k^y), \quad l = 1,2,\ldots,M, \quad k = 1,2,\ldots,N$$

Using the relations (3.8)-(3.11) and (6.5) the matrices $A$ and $D$ can be written as

(6.6) $$A = C.Q + P.H + A_1.(P.Q)$$

(6.7) $$D = P.Q$$

 

## VII. CONCLUSION

In this paper, we have solved eigenvalue problems defined by partial differential equations with constant and variable coefficients, on rectangular or circular domain. We have shown that the proposed method is an interesting versatile technique, which easy to implement on a computer and being capable of producing very accurate approximations of eigenvalue defined by partial eigenvalue problems.

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